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Instanton calculations using dimensional regularisation

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Abstract. Dimensional regularisation is used to calculate the determinant of small oscillations about instanton solutions in $g(\phi^4)_4$ field theories with $g<0$. Principal features of the method are: (i) the known instanton solution of four dimensions may be used even though it is not a solution of the regularised theory; (ii) the minimally subtracted renormalised coupling can be introduced directly. Results are also given for O(n) internal symmetry and conformal invariant ϕ^{2N} theories ($g<0$). The high-order estimates obtained agree with those of Lipatov and Brézin *et al.*

1. Introduction

Tunnelling phenomena in quantum mechanics are readily calculated in the semiclassical limit (small $\hbar$) by the WKB approximation. Although the existence of tunnelling in quantum field theory was recognised by Dyson (1952), progress in calculation was limited by the difficulty of generalising the WKB method from quantum mechanics to quantum field theory.

The formalism for the Green functions of the quantum field as functional integrals of $\exp(iS(\phi)/\hbar)$, where S is the classical action of the field ϕ, indicates the problem. In the classically allowed regions where no tunnelling is taking place, the functional integrals are dominated by field configurations which are solutions of the classical field equations. The field ϕ can then be expanded about the classical solution ϕ_c. If we write $\phi=\phi_c+\hat{\phi}$, then

$$S(\phi)=S(\phi_c)+\tfrac{1}{2}\int \hat{\phi}M\hat{\phi}+O(\hat{\phi}^3);$$

no terms linear in $\hat{\phi}$ appear because ϕ_c is an extremum of S. The terms $O(\hat{\phi}^3)$ give correction of order $\hbar$, so that for small $\hbar$ the Gaussian integral on the field $\hat{\phi}$ yields (for bosons)

$$\int D\phi\,\exp(iS(\phi))=\text{constant}\,\exp(iS(\phi_c)/\hbar)(\det M)^{-1/2}(1+O(\hbar)). \qquad (1)$$

In Feynman graph language, the factor involving the determinant of M comes from one-loop diagrams; the corrections of order $\hbar$ correspond to two or more loops. We need to generalise this semiclassical treatment to the classically forbidden tunnelling region where no classical solutions exist.

The resolution of this problem which has emerged (McLaughlin 1972) is that the dominant field configurations for tunnelling are classical solutions in imaginary time. We remark here only that: (i) solutions of the imaginary time equations may be

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expected in the classically forbidden region because the replacement $t \to -i\tau$ implies $(d\phi/dt)^2 \to -(d\delta/d\tau)^2$ and real solutions of the energy equation

$$-\frac{1}{2}\left(\frac{d\phi_c}{d\tau}\right)^2 + V(\phi_c) = E$$

do exist for $E < V(\phi)$; and (ii) such a solution is an extremum of $S(\phi)$ about which quantum fluctuations can be studied as above. Thus to study tunnelling one works in the Euclidean (imaginary time) field theory, as in statistical mechanics. The solutions of the Euclidean field equations are called instantons.

The aim of this paper is to show that dimensional regularisation ('t Hooft and Veltman 1972, 't Hooft 1973) provides a very powerful tool for controlling the zeros and ultraviolet divergences which arise in instanton calculations. The major part of the paper is concerned with the simplest such calculation in four space–time dimensions: we calculate in the one-loop approximation the imaginary part of the Euclidean Green functions of a massless boson field ϕ with self-interaction $\frac{1}{4}g\phi^4$ $(g < 0)$. This imaginary part, which is of course exponentially small in g, is associated with the tunnelling out of the metastable ground state $\phi = 0$.

Much of the technical content of this paper is contained in Lipatov (1976a, b, 1977a, b) and Brézin *et al* (1977), where information on late terms in the ordinary perturbation in g is obtained essentially by a dispersion relation in the variable g. (We refer the reader also to the earlier papers by Langer (1967), Lam (1968), Bender and Wu (1969, 1971, 1973), Banks *et al* (1973), and to Brézin (1977) for a review of this problem.) This dispersion relation requires assumptions on the absence of singularities in the complex g plane other than the cut along the negative g axis (whose discontinuity is given by the instanton calculation). These assumptions are almost certainly false in four dimensions because of the existence of renormalon singularities ('t Hooft 1977, Lautrup 1977, Parisi 1978). Nevertheless we use the dispersion relation to obtain the late terms for the renormalisation group β function in four dimensions. Our results agree with those of Lipatov (1977a, b) and Brézin *et al* (1977).

The main features and advantages of the dimensional regularisation scheme are the following:

(*a*) The instanton solution ϕ_c of four dimensions is not a solution of the classical field equations in d dimensions. Correspondingly when the field ϕ is expanded about the configuration ϕ_c in the regularised theory, there is a term in the action linear in the field $\hat{\phi}$. However we show that this term can be handled by ordinary perturbation theory. Thus even in d dimensions one expands the field ϕ about the known solution of four dimensions.

(*b*) With this choice of ϕ_c, the determinant of the differential operator M in equation (1) can be calculated exactly in d dimensions by a conformal transformation on to the sphere in $(d+1)$ dimensions (Adler 1972, 1973, Drummond 1975, Fubini 1976, 't Hooft 1976, Jackiw and Rebbi 1976, Lipatov 1977b, Brézin *et al* 1977). Thus for M there exists an $O(d+1)$ formalism even though there is no conformal invariance of the original action in d dimensions.

(*c*) For every continuous symmetry of the action which is broken by a classical solution ϕ_c, there is an eigenfunction of M with zero eigenvalue. For such modes the Gaussian approximation in equation (1) is inadequate and the Gaussian integrals must be replaced by exact integrations; this is done using the method of collective coordinates (Zittartz and Langer 1966, Langer 1967, Gervais and Sakita 1975). In dimensional regularisation all eigenvalues which would be zero in four dimensions become of order $(4-d)$ in d dimensions. However all the collective coordinates of four dimensions must be retained in d dimensions in order that the term linear in $\hat{\phi}$ (see remark (a)) can indeed be treated perturbatively. Further, with 'zero modes' which are in fact of order $(4-d)$, one obtains an explicit demonstration that if the 'zero modes' are removed from the differential operator *on the sphere*, then the Jacobian factor in the transformation to collective coordinates is the norm of the corresponding eigenfunction *on the sphere*.

(d) In d dimensions the product of eigenvalues (in det M) is given in terms of the Reimann ζ function and the simple poles in $(4-d)$ characteristic of dimensional regularisation appear directly from the known singularity of that function. Renormalisation can be made directly by minimal subtraction.

The outline of the paper is as follows. In § 2 we consider the classical solution in four dimensions and justify using it in d dimensions. In § 3 we calculate the small oscillations determinant in d dimensions and isolate the pole term and finite part ($d \to 4$). In § 4 we introduce the collective coordinates for the zero modes; the expression for the imaginary part in terms of the bare coupling is given in equation (37). Renormalisation is performed in § 5 and the result in terms of the renormalised (minimally subtracted) running coupling constant is given in equation (42). The late terms of the β function are obtained in § 6 and compared with Lipatov (1977a, b) and Brézin *et al* (1977). Two appendices generalise the results to include O(n) internal symmetry and conformal invariant $g\phi^{2N}$ theories ($g<0$).

2. Classical solution and the perturbation about it

We start from the Euclidean Green functions

$$G^{(2M)} = \frac{\int D\phi\phi(x_1)\ldots\phi(x_{2M})\exp(-H(\phi))}{\int D\phi\exp(-H(\phi))} \tag{2}$$

where

$$H(\phi) = \int d^dx[\tfrac{1}{2}\phi(-\nabla^2)\phi + \tfrac{1}{4}g\phi^4] \tag{3}$$

measures the Euclidean action. The field equations are

$$\nabla^2\phi = g\phi^3 \tag{4}$$

for which, for $g<0$, and in four dimensions there are Emden solutions (Emden 1907)

$$\phi = \pm\phi_c \tag{5}$$

where

$$\phi_c = \left(\frac{8}{-g}\right)^{1/2}\frac{\lambda}{1+\lambda^2(x-x_0)^2}. \tag{6}$$

The parameters x_0 and λ characterise the position and scale size of the instanton ϕ_c, and are associated with the translation and dilatation invariance of H.

When the fields in (2) are expanded about ϕ_c we obtain exponentially small imaginary parts in both numerator and denominator. In order to isolate the imaginary part of the numerator, we define

$$Z^{(2M)} = \frac{\int D\phi\phi(x_1)\ldots\phi(x_{2M})\exp(-H(\phi))}{\int D\phi\,\exp(-H_0(\phi))} \tag{7}$$

where H_0 is a free Hamiltonian ($g = 0$); the numerator is to be expanded about $\phi = \phi_c$, the denominator about $\phi = 0$.

The theory is regularised by working in d dimensions. The continuation to general d is defined by the following results.

(*a*) Integrals over all space can be translated and scaled as usual and

$$\int_{-\infty}^{\infty} d^d x\, f(r) = S_d \int_0^{\infty} dr\, r^{d-1} f(r), \qquad r = |x| \tag{8}$$

where $S_d = 2\pi^{d/2}/\Gamma(d/2)$ is the surface of the unit sphere in d dimensions.

(*b*)

$$\nabla^2 = \frac{d^2}{dr^2} + \frac{d-1}{r}\frac{d}{dr} - \frac{L^2}{r^2} \tag{9}$$

where

$$L^2 = -\frac{1}{2}\sum_{i,j}\left(x_i\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial x_i}\right)^2.$$

(*c*) Symmetric traceless tensors $f^{(l)} = u_{i_1 i_2 \ldots i_l} x_{i_1} x_{i_2} \ldots x_{i_l}$ ($u_{iii_3\ldots i_l} = 0$) are spherical harmonic eigenfunctions of L^2 with eigenvalue $l(l+d-2)$:

$$L^2 f^{(l)} = l(l+d-2) f^{(l)}, \tag{10}$$

and have degeneracy

$$\nu_l(d) = \frac{(2l+d-2)\Gamma(l+d-2)}{\Gamma(d-1)\Gamma(l+1)}. \tag{11}$$

With this definition of ∇^2 in d dimensions we can now proceed to the Gaussian approximation of expression (7). Writing $\phi = \phi_c + \hat{\phi}$, we have

$$H(\phi) = H(\phi_c) + \tfrac{1}{2}\int d^d x\, \hat{\phi} M \hat{\phi} - \frac{\epsilon 4(2)^{1/2}}{(-g)^{1/2}} \int d^d x \frac{\lambda^3}{[1+\lambda^2(x-x_0)^2]^2}\hat{\phi}$$
$$- (-8g)^{1/2} \int d^d x \frac{\lambda}{1+\lambda^2(x-x_0)^2}\hat{\phi}^3 + \frac{g}{4}\int d^d x\, \hat{\phi}^4. \tag{12}$$

$H(\phi_c)$ must be evaluated correct to first order in $\epsilon = 4-d$ because after one-loop renormalisation $1/g \to (1/g_R) + O(\epsilon^{-1})$ (see § 5). We obtain

$$H(\phi_c) = -\frac{\lambda^{\epsilon}}{g}\frac{8\pi^2}{3}[1 - \tfrac{1}{2}\epsilon(2 + \ln\pi + \gamma) + O(\epsilon^2)]. \tag{13}$$

The quadratic form M is

$$M = -\nabla^2 - \frac{24\lambda^2}{[1+\lambda^2(x-x_0)^2]^2}. \tag{14}$$

The linear term (in $\hat{\phi}$) in equation (12) exists because ϕ_c is not a solution of the field equations in d dimensions ($d \neq 4$). Normally if one has a term linear in the field in the Hamiltonian, the functional integral must be brought into a proper Gaussian form by translating the field to eliminate the linear term. This is equivalent in this case to solving the non-linear equation (4) exactly. However in our particular case, the linear term in (12) can be handled by straightforward perturbation theory.

To establish this claim, consider first connected tree diagrams with $\hat{\phi}$ insertions. Since the coefficient of $\hat{\phi}$ in equation (12) is of order $\epsilon/(-g)^{1/2}$, inspection shows that a tree diagram with L $\hat{\phi}$ insertions is of order ϵ^L/g, where L is always greater than one. The contributions of all tree diagrams give the exponential of the connected diagrams, and hence these contributions or order ϵ^L/g are to be added to $H(\phi_c)$ in expression (13). However as remarked there, terms of order ϵ^2/g are negligible for our calculation even after renormalisation. Hence all these tree diagrams contribute only to a higher-order calculation. Implicit in these remarks is the assumption that the propagators, which are the inverse of M in equation (14), contain no eigenvalues of order $1/\epsilon$. Therefore the small oscillations field $\hat{\phi}$ must contain no modes of M with eigenvalues of order ϵ, i.e. the zero modes of four dimensions must still be handled in d dimensions by collective coordinates, and excluded from the small oscillations $\hat{\phi}$.

Next consider one-loop diagrams, where a factor of $1/\epsilon$ can appear from the ultraviolet divergences. The only possible candidate is shown in figure 1. The $1/\epsilon$ divergence of the loop gives a contribution of order 1. The calculation of this diagram is given in appendix 1, and should be comprehensible after §§ 3 and 4. The value of the graph is $-\frac{9}{2}$ and hence it contributes a factor

$$c_1 = e^{-9/2} \tag{15}$$

to the coefficient of the imaginary part of $Z^{(2M)}$ (equation (7)).

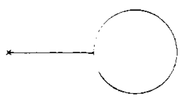

Figure 1. The only graph involving the linear term in the Hamiltonian (12) which contributes at this order.

3. Determinant of small oscillations

In this section we calculate the ratio $\det M/\det M_0$, as required in expression (1). Here M is as in equation (14) and M_0 is the operator $-\nabla^2$, from the denominator of expression (7). A direct calculation involves finding the bound state spectrum and phase shifts of M. In fact the only information on the spectrum of M is contained in the zero modes arising from the spontaneous symmetry breaking by ϕ_c. Specifically, in four dimensions

$$\frac{\partial\phi_c}{\partial\lambda} = \left(\frac{8}{-g}\right)^{1/2} \frac{1-\lambda^2(x-x_0)^2}{[1+\lambda^2(x-x_0)^2]^2} \tag{16a}$$

and

$$\frac{\partial\phi_c}{\partial x_0^\mu} = -\left(\frac{8}{-g}\right)^{1/2} \frac{2\lambda^2(x-x_0)_\mu}{[1+\lambda^2(x-x_0)^2]^2} \tag{16b}$$

are eigenfunctions of M with zero eigenvalue. The eigenfunction (16*b*) corresponds to $l = 1$ and is fourfold degenerate. The fact that the $l = 0$ eigenfunction (16*a*) has a node at $(x - x_0)^2 = 1/\lambda^2$ indicates that there is a lower energy $l = 0$ eigenfunction, i.e. a bound state.

The role of a single bound state is best appreciated by considering a model integral

$$Z(g) = \int \mathrm{d}\phi \, \exp[-(\tfrac{1}{2}\phi^2 + \tfrac{1}{4}g\phi^4)]. \tag{17}$$

For Re $g > 0$, $Z(g)$ is defined by a contour integral along the real ϕ axis. $Z(g)$ is defined for other values of arg g by standard analytic continuation by rotation on the contour of ϕ integration. The cases arg $g = \pm\pi$ are defined by contours which have for large $|\phi|$, arg $\phi = \mp\pi/4$. The steepest descent evaluation of these contour integrals for negative g gives imaginary parts dominated by the 'instanton' saddle points $\phi_c = \pm(-g)^{-1/2}$, as shown in figure 2. The negative 'eigenvalue' associated with the 'instanton' merely implies that the corresponding integration contour of steepest descent goes into the complex plane. For future reference note that each saddle point gives only half a Gaussian integration and that if the analytic continuation in g of the functional integral follows that of the toy, then

$$\mathrm{sgn}[\mathrm{Im}\, Z^{(2M)}(\arg g = \pm\pi)] = \mp 1. \tag{18}$$

(We are grateful to M Stone for discussions on this point.)

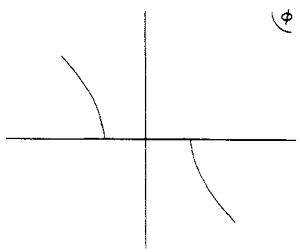

Figure 2. The steepest descent contour for the function $Z(g; \arg g = \pi)$ in equation (17); the dominant part comes from the saddle point at $\phi = 0$; the imaginary part is dominated by the instanton saddle points $\phi = \pm 1/(-g)^{1/2}$.

In addition to these zero modes and bound state in four dimensions, one knows that the determinant represents the one-loop diagrams in the presence of an instanton, and will have the conventional ultraviolet divergences from the product of eigenvalues of high angular momentum. It is to control this divergence that we work in d dimensions. Of course ϕ_c is then not a solution of the field equations. In fact it *is* the exact solution of a field equation in d dimensions with an external source term

$$\epsilon\left(\frac{8}{-g}\right)^{1/2} \frac{2\lambda^3}{[1 + \lambda^2(x - x_0)^2]^2}$$

which breaks *all* symmetries by order ϵ. Hence in d dimensions all the 'zero modes' have eigenvalues of order ϵ.

In order to calculate det M/det M_0 we must know eigenvalues (and phase shifts in the continuum) of the differential operator M. However its general eigenfunctions are not known. This problem is avoided by exploiting the dynamical $\mathrm{O}(d + 1)$ invariance of the quadratic form. After translating x by x_0 and rescaling by λ, we define fields Φ on the unit hypersphere in $(d+1)$ dimensions:

$$\Phi = \kappa^{1-(d/2)}\phi \tag{19}$$

where

$$\kappa = 2/(1+x^2). \tag{20}$$

The volume element d^dx is replaced by $d\Omega\,\kappa^{-d}$ where $d\Omega$ is the surface element of the unit sphere in $(d+1)$ dimensions. The differential operator ∇^2 is replaced by

$$V_0 = -L^2 - \tfrac{1}{4}d(d-2) \tag{21}$$

where L^2 is the total angular momentum operator in $(d+1)$ dimensions (cf equations (9), (10) and (11)). The Hamiltonian (12) now has the form

$$H = H(\phi_c) + \tfrac{1}{2}\int d\Omega\,\hat{\Phi}(-V_0-6)\hat{\Phi} - \epsilon(2/-g)^{1/2}\int d\Omega\,\kappa^{-1+(\epsilon/2)}\hat{\Phi}$$
$$-(-2g)^{1/2}\int d\Omega\,\kappa^{-\epsilon/2}\hat{\Phi}^3 + \tfrac{1}{4}g\int d\Omega\,\kappa^{-\epsilon}\hat{\Phi}^4. \tag{22}$$

The quadratic form in this expression is invariant under $O(d+1)$ rotations; its eigenfunctions are the spherical harmonics. Using equations (9), (10), (11) and (21) we obtain for the determinant of small oscillations

$$\left(\frac{\det M}{\det M_0}\right)^{-1/2} = \prod_{L=0}^{\infty}\left(\frac{(L+\frac{1}{2}d-3)(L+\frac{1}{2}d+2)}{(L+\frac{1}{2}d-1)(L+\frac{1}{2}d)}\right)^{-\frac{1}{2}\nu_L(d+1)}. \tag{23}$$

In the numerator on the right-hand side we note the bound state $L=0$ and the $(d+1)$ 'zero modes' $(L=1)$ with eigenvalue

$$E = -\tfrac{5}{2}\epsilon(1+O(\epsilon)). \tag{24}$$

Extracting this factor from the product, the right-hand side of (23) becomes

$$c_2 = 2^2 3^{5/2}\exp -\frac{1}{2}\sum_{L=2}^{\infty}\frac{\Gamma(L+3-\epsilon)(2L+3-\epsilon)}{\Gamma(4-\epsilon)\Gamma(L+1)}\ln\left(\frac{(L-1-\frac{1}{2}\epsilon)(L+4-\frac{1}{2}\epsilon)}{(L+1-\frac{1}{2}\epsilon)(L+2-\frac{1}{2}\epsilon)}\right)(1+O(\epsilon))\Big]. \tag{25}$$

The $(-1)^{1/2}$ from the bound state has been omitted since it generates only the imaginary part of $Z^{(2M)}$ (cf equation (18)).

Since $\Gamma(L+3-\epsilon)(2L+3-\epsilon)/\Gamma(L+1) \sim L^{3-\epsilon}$ for L large, the sum diverges as $\epsilon \to 0$. To pick out the simple pole and finite part as $d \to 4$ we shift the summation variable independently for each of the four logarithmic factors so that the sums may be written in the form

$$\sum_{L=2}^{\infty} L^p \frac{\Gamma(L-\epsilon)}{\Gamma(L)}\ln(L-\tfrac{1}{2}\epsilon)$$

where p is an integer. Using the asymptotic expansion

$$\frac{\Gamma(L-\epsilon)}{\Gamma(L)} = L^{-\epsilon}\left[1+\epsilon\left(\frac{1}{2L}+\frac{1}{12L^2}+O\left(\frac{1}{L^4}\right)\right)+\epsilon^2\left(\frac{1}{2L}+\frac{3}{8L^2}+O\left(\frac{1}{L^3}\right)\right)+O(\epsilon^3)\right] \tag{26}$$

we find that the sums reduce to $\Sigma_{L=2}^{\infty} L^{1-\epsilon}\ln L$ or $\Sigma_{L=2}^{\infty} L^{-1-\epsilon}\ln L$. These are respectively $-\zeta'(-1+\epsilon)$ and $-\zeta'(1+\epsilon)$, where $\zeta'(s)$ is the derivative of the Riemann ζ

function. The only properties required of the ζ function are that it has a simple pole at $s=1$ with residue 1, and obeys (Erdélyi 1955)

$$\zeta(s)=2^s\pi^{s-1}\sin(\tfrac{1}{2}\pi s)\Gamma(1-s)\zeta(1-s).$$

A tedious but straightforward calculation yields

$$c_2=(2\pi)^{-1/2}5^{5/2}\exp\left(\frac{3}{\epsilon}+\frac{3}{4}-\frac{7}{2}\gamma+\frac{3}{\pi^2}\zeta'(2)+O(\epsilon)\right) \tag{27}$$

where $\gamma=0\cdot577\,215\,665\ldots$ is Euler's constant and $\zeta'(2)=-0\cdot937\,548\,254\ldots$. The simple pole in ϵ is directly related to the ultraviolet divergence of one-loop diagrams and is removed by a conventional coupling constant renormalisation as discussed in § 5.

4. Introduction of collective coordinates

In this section we discuss the use of collective coordinates (Zittartz and Langer 1966, Langer 1967, Gervais and Sakita 1975) to deal with the $(d+1)$ zero modes of the operator M.

The existence of these zero modes in four dimensions means that the inverse of the quadratic form does not exist and so we cannot define a propagator.

Even in d dimensions the 'zero modes' must be excluded from the field according to the arguments of § 2. This problem is resolved by excluding the zero mode eigenfunctions from $\hat{\phi}$ and spanning the space of these eigenfunctions by allowing λ and x_0 to vary, and integrating on them. Thus we write

$$\phi(x)=\phi_c(x;\lambda,x_0)+\sum_n a_n\phi_n(x;\lambda,x_0) \tag{28}$$

regarding $\{\lambda, x_0, a_n\}$ as new variables replacing $\phi(x)$ and choosing $\{\phi_n\}$ to be normalised eigenfunctions of M corresponding to the non-zero modes. (ϕ_c is the function in equation (6); ϕ_n depends on λ and x_0 because M does.) Thus the functional measure $D\phi$ is replaced by $J^M\,d^dx_0\,d\lambda\,\Pi_n\,da_n$ where J^M is the Jacobian of the transformation. (The superscript M refers to its association with the differential operator M.)

Performing the Gaussian integral leads to a factor

$$\frac{1}{(2\pi)^{(d+1)/2}}\left(\frac{\det\tilde{M}}{\det M_0}\right)^{-1/2}$$

where the tilde indicates that the zero modes have been extracted and the factor $(2\pi)^{-(d+1)/2}$ comes from the fact that there are $(d+1)$ more Gaussian integrals in the denominator than in the numerator.

Apart from the integrals on λ and x_0, it remains only to calculate the Jacobian. This is given by

$$J^M=\left[\int d^dx\,\phi_\lambda(x)\phi_\lambda(x)\left(\frac{1}{d}\int d^dx\,\phi_\mu(x)\phi_\mu(x)\right)^d\right]^{1/2}(1+O(\epsilon,g)) \tag{29}$$

where $\phi_\lambda\equiv\partial\phi_c/\partial\lambda$ and $\phi_\mu\equiv\partial\phi_c/\partial x_0^\mu$ (equation (16)) are just, in four dimensions, the eigenfunctions of M with zero eigenvalue.

Note, however, that the norm of the dilatation eigenfunction is logarithmically divergent in four dimensions. This indicates the presence of other factors which will cancel off this spurious infrared divergence. These factors come from the small oscillations determinant. To see this recall that in § 3 we calculated $c_2 = (\det \tilde{V}/\det V_0)^{-1/2}$ (equation (27)) *not* $(\det \tilde{M}/\det M_0)^{-1/2}$. In fact

$$\left|\frac{\det \tilde{M}}{\det M_0}\right|^{-1/2} = \left(\frac{E^M_\lambda}{E^V_\lambda}\right)^{1/2}\left(\frac{E^M_\mu}{E^V_\mu}\right)^{d/2}\left|\frac{\det \tilde{V}}{\det V_0}\right|^{-1/2} \tag{30}$$

where E^M_λ, E^M_μ and E^V_λ, E^V_μ are the regularised eigenvalues of M and V respectively corresponding to the dilatation and translation eigenfunctions. Thus we have an extra factor $(E^M_\lambda/E^V_\lambda)^{1/2}(E^M_\mu/E^V_\mu)^{d/2}$ to take into account. E^V_λ and E^V_μ are given by equation (24) whereas E^M_λ and E^M_μ can be calculated to lowest order in ϵ in straightforward perturbation theory:

$$E^M_\lambda = \frac{\int \mathrm{d}^d x\, \phi_\lambda(x)(M_d - M_4)\phi_\lambda(x)}{\int \mathrm{d}^d x\, \phi_\lambda(x)\phi_\lambda(x)} \tag{31}$$

and

$$E^M_\mu = \frac{\int \mathrm{d}^d x\, \phi_\mu(x)(M_d - M_4)\phi_\mu(x)}{\int \mathrm{d}^d x\, \phi_\mu(x)\phi_\mu(x)}. \tag{32}$$

We see that the norms in the Jacobian (29) are cancelled by the denominators in (31) and (32) and we are left with a finite result as $\epsilon \to 0$.

A more instructive way of proceeding is to notice that since $V\Phi = \kappa^{-1-(d/2)}M\phi$ where $\Phi = \kappa^{1-(d/2)}\phi$, then $V\Phi = E^V\Phi$ implies $M\phi = 4\lambda^2 E^V[1+\lambda^2(x-x_0)^2]^{-2}\phi$. Thus

$$E^V_\lambda = \frac{\int \mathrm{d}^d x\, \phi_\lambda(x)(M_d - M_4)\phi_\lambda(x)}{\int \mathrm{d}^d x\, \phi_\lambda(x) 4\lambda^2[1+\lambda^2(x-x_0)^2]^{-2}\phi_\lambda(x)} \tag{33}$$

and

$$E^V_\mu = \frac{\int \mathrm{d}^d x\, \phi_\mu(x)(M_d - M_4)\phi_\mu(x)}{\int \mathrm{d}^d x\, \phi_\mu(x) 4\lambda^2[1+\lambda^2(x-x_0)^2]^{-2}\phi_\mu(x)} \tag{34}$$

to lowest order in ϵ.

Combining equations (28)–(34), we have

$$J^M(2\pi)^{-(d+1)/2}\left(\frac{\det \tilde{M}}{\det M_0}\right)^{-1/2} = J^V(2\pi)^{-(d+1)/2}\left(\frac{\det \tilde{V}}{\det V_0}\right)^{-1/2} \tag{35}$$

where J^V is the Jacobian factor with the 'compact norm'

$$\begin{aligned} J^V &= \bigg[\int \mathrm{d}^d x\, \phi_\lambda(x) 4\lambda^2[1+\lambda^2(x-x_0)^2]^{-2}\phi_\lambda(x) \\ &\qquad \times\left(\frac{1}{d}\int \mathrm{d}^d x\, \phi_\lambda(x) 4\lambda^2[1+\lambda^2(x-x_0)^2]^{-2}\phi_\mu(x)\right)^d\bigg]^{1/2}(1+\mathrm{O}(\epsilon, g)) \\ &= \lambda^{d-1}\left(-\frac{16\pi^2\lambda^\epsilon}{15g}\right)^{(d+1)/2}(1+\mathrm{O}(\epsilon, g)). \end{aligned} \tag{36}$$

Collecting all the factors from equations (13), (15), (18), (27), (35) and (36), we have

$$\operatorname*{Im}_{(\arg g = \pi)} Z^{(2M)}(x_1, \ldots, x_{2M}) = -\int \mathrm{d}\lambda\, \mathrm{d}^d x_0\, c_1 c_2 (2\pi)^{-(d+1)/2} \exp(-H(\phi_c)) J^V \prod_{i=1}^{2M} \phi_c(x_i)(1+\mathrm{O}(\epsilon, g))).$$

Taking the Fourier transform, extracting $(2\pi)^d\delta(\Sigma_i q_i)$, and amputating the external leg factors gives for the vertex function

$$\operatorname*{Im}_{(\arg g=\pi)}\Gamma^{(2M)}(q_i) = -C_b\int_0^\infty \frac{d\lambda}{\lambda}\lambda^{d-M(d-2)}\left(-\frac{\lambda^\epsilon 8\pi^2}{3g}\right)^{(d+1+2M)/2}$$

$$\times\exp\left[\frac{\lambda^\epsilon}{g}\frac{8\pi^2}{3}\left(1-\frac{\epsilon}{2}(\ln\pi+\gamma+2)\right)\right]\prod_{i=1}^{2M}(q_i^2/\lambda^2\tilde{\phi}(q_i/\lambda))(1+O(\epsilon,g)). \tag{37}$$

In this expression, $\tilde{\phi}$ is obtained from the Fourier transform of ϕ_c after a factor $(-\lambda^\epsilon 8\pi^2/3g)^{1/2}$ has been removed:

$$\tilde{\phi}(q) = \frac{3^{1/2}}{\pi}\int d^dx\, e^{iq.x}(1+x^2)^{-1} = 2^{d/2}\pi^{(d-2)/2}3^{1/2}|q|^{1-(d/2)}K_{(d/2)-1}(|q|) \tag{38}$$

where K is a modified Bessel function. The coefficient C_b is given by

$$C_b = 2^{-1/2}\pi^{-3}\exp\left(\frac{3}{\epsilon}+\frac{3\zeta'(2)}{\pi^2}-\frac{7}{2}\gamma-\frac{15}{4}\right). \tag{39}$$

Equations (37) to (39) contain the final result for the bare theory.

Note that although there are two saddle points $\phi = \pm\phi_c$, each contributes only half a Gaussian integral according to the discussion in § 2; thus the bound state mode effectively contributes a full Gaussian integral. Also, after removal of the external legs, the imaginary part at this order is indeed one-particle irreducible as there are no poles in any momentum variable. Finally note that the imaginary part from the denominator in equation (2) (cf. equation (7)) is of higher order in g and can be neglected.

5. Renormalisation

In the massless theory which we are considering, the divergences in perturbation theory are removed by a coupling constant and wavefunction renormalisation. We have

$$\Gamma_R^{(2M)} = Z^M\Gamma^{(2M)}$$

where $Z = 1+O(g^2)$. For minimal subtraction ('t Hooft 1973)

$$\mu_0^{-\epsilon}g = g_R(\mu_0)+\frac{9}{8\pi^2\epsilon}g_R^2(\mu_0)+O(g_R^3) \tag{40}$$

where $g_R(\mu_0)$ is the renormalised coupling at the momentum scale μ_0.

For the one-loop result in expression (37), we require only one-loop coupling constant renormalisation; the wavefunction renormalisation Z contributes to Im Γ only at the three-loop level. In the expression (37), we see that the combination λ^ϵ/g naturally appears in the exponential. Thus the effective coupling for instantons of scale size $1/\lambda$ is the renormalised coupling at momentum scale λ. Writing

$$\frac{\lambda^\epsilon}{g} = \frac{1}{g_R(\lambda)}-\frac{9}{8\pi^2\epsilon} \tag{41}$$

we obtain the renormalised imaginary part

$$\operatorname*{Im}_{(\arg g=\pi)} \Gamma_{\mathrm{R}}^{(2M)}(q_i) = -C_{\mathrm{R}} \int_0^\infty \frac{\mathrm{d}\lambda}{\lambda} \lambda^{d-M(d-2)} \left(-\frac{8\pi^2}{3g_{\mathrm{R}}(\lambda)}\right)^{(d+1+2M)/2} \prod_{i=1}^{2M} [q_i^2/\lambda^2 \tilde{\phi}(q_i/\lambda)]$$
$$\times \exp\left[\frac{1}{g_{\mathrm{R}}(\lambda)} \frac{8\pi^2}{3}\left(1-\frac{\epsilon}{2}(\ln \pi + \gamma + 2)\right)\right](1+\mathrm{O}(\epsilon, g)) \tag{42}$$

where

$$C_{\mathrm{R}} = C_{\mathrm{b}} \exp\left(-\frac{3}{\epsilon} + \frac{3}{2}(\ln \pi + \gamma + 2) + \mathrm{O}(\epsilon)\right)$$
$$= 2^{-1/2}\pi^{-3/2} \exp\left(\frac{3}{\pi^2}\zeta'(2) - 2\gamma - \frac{3}{4}\right). \tag{43}$$

From equation (41), the λ dependence of $g_{\mathrm{R}}(\lambda)$ may be extracted by rewriting it in terms of a renormalised coupling at a fixed momentum scale μ_0:

$$\frac{1}{g_{\mathrm{R}}(\lambda)} = \frac{(\lambda/\mu_0)^\epsilon}{g_{\mathrm{R}}(\mu_0)} + \frac{9}{8\pi^2\epsilon}\left[1 - \left(\frac{\lambda}{\mu_0}\right)^\epsilon\right]. \tag{44}$$

Thus from the asymptotic freedom of the theory (in particular for $d = 4$ since $g < 0$), we see that the integral in (42) converges for large λ, for any $M \geqslant 1$. For small λ convergence is guaranteed by the exponential decrease of $\tilde{\phi}$ (see equation (38)). Equations (42) to (44) contain the final result for the renormalised imaginary part.

6. High-order estimates

The existence of an imaginary part of vertex functions for $g < 0$ implies that they have a cut in the g plane which can be placed along the negative g axis. The fact that this singularity extends up to the origin implies that the perturbation expansion in g has zero radius of convergence. In the absence of stronger singularities (from other sources) at the origin in the g plane, one can obtain the leading behaviour of the late terms in the perturbation expansion by means of a dispersion relation in g, with the contour shown in figure 3:

$$\Gamma^{(2M)}(q_i; g) = \frac{1}{\pi}\int_{-\infty}^0 \frac{\mathrm{d}g'}{g'-g} \operatorname{Im} \Gamma^{(2M)}(q_i; g'(\arg g' = \pi))$$
$$\sim \sum_K g^K \frac{1}{\pi} \int_{-\infty}^0 \frac{\mathrm{d}g'}{(g')^{K+1}} \operatorname{Im}\Gamma^{(2M)}(q_i; g'(\arg g' = \pi)). \tag{45}$$

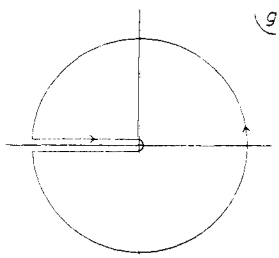

Figure 3. Contour integral involved in the dispersion relation (45).

(We have assumed no subtractions are required, and have discarded the contour at infinity; the estimates for late terms are unaffected if any *finite* number of subtractions is required.)

In general in four dimensions, however, other singularities do exist. They are the singularities due the the Landau ghost (the so called renormalon effects: Landau 1955, 't Hooft 1977, Lautrup 1977, Parisi 1978). It is believed that these singularities are present in the vertex functions and in the renormalisation group β function unless the latter is expressed in terms of the minimally subtracted coupling constant. They are believed to be absent for $d<4$, in particular for ϵ expansions. Since we have calculated the imaginary part using dimensional regularisation, we are in an ideal position to obtain high-order estimates for quantities which are believed to be free of renormalon singularities. These results will be published elsewhere. In this paper we restrict ourselves to repeating the high-order estimates for the β function in four dimensions for a renormalised coupling constant defined at a symmetry point μ (Lipatov 1977a, b, Brézin *et al* 1977).

To be specific, we write a dispersion relation in the one-loop renormalised (minimally subtracted) coupling $g_R(\mu_0)$. The *full* renormalised coupling $\bar{g}_R(\mu)$ defined at the symmetry point μ is given by

$$\bar{g}_R(\mu)\mu^{\epsilon}=-\tfrac{1}{6}\Gamma_R^{(4)}(q_i)|_{q_i\cdot q_j=\frac{1}{3}\mu^2(4\delta_{ij}-1)}. \tag{46}$$

(The factor $-\frac{1}{6}$ is from our normalisation conventions.) The β function is given by $\mu\, \mathrm{d}\bar{g}_R(\mu)/\mathrm{d}\mu$ at fixed bare theory, or equivalently at fixed μ_0 and $g_R(\mu_0)$. Thus using equations (42), (45) and (46) the coefficient of $(g_R(\mu_0))^K$ in this β function is given by

$$\tilde{\beta}_K=\frac{C_R}{6\pi}\int_{-\infty}^{0}\frac{\mathrm{d}g}{g^{K+1}}\int_0^{\infty}\frac{\mathrm{d}\lambda}{\lambda}\left[-\frac{8\pi^2}{3g}+3\ln\left(\frac{\lambda}{\mu_0}\right)\right]^{9/2}$$
$$\times\exp\left[\frac{8\pi^2}{3g}\right]\left(\frac{\lambda}{\mu_0}\right)^{-3}\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\left(\frac{\mu^2}{\lambda^2}\tilde{\phi}\left(\frac{\mu}{\lambda}\right)\right)^4(1+\mathrm{O}(g)) \tag{47}$$

where we have taken the limit $d\to4$ since it is finite, and g means the one-loop renormalised $g_R(\mu_0)$. The integral on g gives trivially a Γ function; the correction of order g in $\mathrm{Im}\,\Gamma$ gives contributions down by $\mathrm{O}(K^{-1})$ and are negligible. The replacement $x=\mu/\lambda$ in the λ integral and integration by parts gives

$$\tilde{\beta}_K=\frac{C_R}{2\pi}\left(-\frac{3}{8\pi^2}\right)^K\Gamma(K+\tfrac{9}{2})\left(\frac{\mu}{\mu_0}\right)^{-3}\int_0^{\infty}\mathrm{d}x\,x^{10}(\tilde{\varphi}(x))^4(1+\mathrm{O}(K^{-1})). \tag{48}$$

Recall that this is the coefficient of $(g_R(\mu_0))^K$ in the β function. In order to obtain the β function in terms of the full renormalised coupling $\bar{g}_R(\mu)$, we must make the substitution

$$g_R(\mu_0)=\bar{g}_R(\mu)+\alpha(\bar{g}_R(\mu))^2+\mathrm{O}(\bar{g}^3). \tag{49}$$

(Since β starts at $\mathrm{O}(g^2)$ in four dimensions, the higher-order terms in this expression are irrelevant.)

Simple algebra then gives the coefficient of $(\bar{g}_R(\mu))^K$ in β to be

$$\beta_K=\mathrm{e}^{-8\pi^2\alpha/3}\tilde{\beta}_K(1+\mathrm{O}(K^{-1})). \tag{50}$$

The coefficient α comes from straightforward perturbation theory. Evaluating the usual one-loop diagram at the symmetry point μ gives, using the definition (46),

$$\mu^{\epsilon}\bar{g}_R(\mu)=g-9(8\pi^2\epsilon)^{-1}g^2\mu^{-\epsilon}[1+\epsilon(\tfrac{1}{2}\ln\pi+\tfrac{1}{2}\ln3-\tfrac{1}{2}\gamma+1)+\mathrm{O}(\epsilon^2)]+\mathrm{O}(g^2) \tag{51}$$

where here g means the bare coupling constant. Eliminating g between equations (41) and (51) gives in four dimensions

$$\alpha = 9(8\pi^2)^{-1}[\tfrac{1}{2}\ln\pi + \tfrac{1}{2}\ln 3 - \tfrac{1}{2}\gamma + 1 - \ln(\mu/\mu_0)]. \tag{52}$$

If we substitute this value of α into equation (50), and use the explicit forms (38) for $\tilde{\phi}$ and (43) for C_R in the expression (48) for β_K, we obtain the required result

$$\beta_K = 3^{1/2}2^{13/2}\exp\left(\frac{3}{\pi^2}\zeta'(2) - \tfrac{15}{4} - \tfrac{1}{2}\gamma\right)\int_0^\infty dx\, x^6(K_1(x))^4\left(-\frac{3}{8\pi^2}\right)^K \Gamma(K+\tfrac{9}{2})[1+O(K^{-1})]. \tag{53}$$

Up to the convention of the normalisation of coupling constant ($g/4!$ or $g/4$) this result is identical to that obtained by Lipatov (1977a, b) and Brézin *et al* (1977). To verify this one can evaluate the sum

$$\sum = \exp\left(-\tfrac{1}{12}\sum_{l=2}^{\infty}(l+1)(l+2)(2l+3)\left[\ln\left(1-\frac{6}{(l+1)(l+2)}\right) + \frac{6}{(l+1)(l+2)} + \frac{18}{(l+1)^2(l+2)^2}\right]\right)$$

(from, e.g., the results section (b)(ii) of Brézin *et al* for $n = 1$). This can be done following the method of 't Hooft (1976) by placing an integer cut-off Λ on the sum on l and evaluating each of the three terms in the sum independently for large Λ. For the first term the trick is to change the summation variable independently for each of the four logarithmic factors so that each is of the form $\Sigma\, P(l) \ln l$ where $P(l)$ is a polynomial. Apart from 'end terms' from changing the summation variable one ends up with

$$\sum_{l=2}^{\Lambda} l \ln l = \tfrac{1}{2}\Lambda^2\ln\Lambda + \tfrac{1}{2}\Lambda\ln\Lambda + \tfrac{1}{12}\ln\Lambda - \tfrac{1}{4}\Lambda^2 + \tfrac{1}{12}(\ln 2\pi + \gamma) - (\zeta'(2)/2\pi^2).$$

The Λ dependence cancels as it must and the final result is

$$\sum = 2^{-5/2}3^{-5/2}5^{5/2}\pi^{-1/2}\exp(3\pi^{-2}\zeta'(2) + \tfrac{17}{4} - \tfrac{7}{2}\gamma).$$

Substituting this expression into the result of Brézin *et al* reproduces equation (53) up to a factor 6 for the different choice of coupling normalisations.

7. Conclusions

In this paper we have calculated the imaginary part (of vertex functions) associated with tunnelling in field theories with negative coupling, in the semiclassical (small g) approximation. The new feature of this paper is the use of dimensional regularisation to control the ultraviolet divergences of four dimensions. A major aspect is that even in the regularised theory ($d \neq 4$) it is adequate to use the instanton solution of four dimensions. Because this field configuration is not a solution in $d(\neq 4)$ dimensions, one may obtain additional insight into the Jacobian factor when collective coordinates are introduced. The integration over instanton positions x_0^μ restores translation invariance. Another advantage is the direct appearance of the dimensionally regularised, renormalised, running coupling constant; instantons of scale size $1/\lambda$ have an effective coupling $g_R(\lambda)$. The asymptotic freedom of the theory in $d=4$ for $g<0$ implies the convergence over instanton scale sizes for $\lambda\to\infty$, and for fixed external momenta there is exponential convergence for $\lambda\to 0$.

Via a dispersion relation in g we have obtained the late terms in the β function, in four dimensions, for the renormalised coupling defined at a symmetry point μ. The result, generalised to $O(n)$ internal symmetry in appendix 2, agrees analytically with the calculation of Lipatov (1977a, b) and Brézin *et al* (1977), although there is minor numerical disagreement with the results quoted in Brézin *et al* for $n\neq 1$.

The generalisation to conformal invariant $g\phi^{2N}$ theories is given in appendix 3.

A final, perhaps unimportant feature, which we relegated for technical reasons to appendix 1, is the evaluation of the contribution of the extra diagram, figure 1. The calculation can be completely controlled by working on the sphere with the Hamiltonian (22). It is this result which ensures agreement with the calculations of Lipatov and Brézin *et al*. We feel the graph should also be directly calculable in flat space using the Hamiltonian (12); we have not been able to elucidate the discrepancy discussed in appendix 1.

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Appendix 1

Here we evaluate the contribution from the linear term in the Hamiltonian (12). As discussed in § 2 there is only one graph giving a contribution at this order; it is shown in figure 1. Since lines in this diagram represent the propagator in the presence of an instanton, it is much simpler to use the Hamiltonian on the sphere, expression (22), where the propagator can be written explicitly in terms of the spherical harmonics $Y_L^\alpha(\eta)$ in $(d+1)$ dimensions; the label L is the principal quantum number and η are the usual coordinates on the (unit) sphere (in $d+1$ dimensions)

$$\eta^\mu=2x^\mu/(1+x^2)\qquad \eta^{d+1}=(1-x^2)/(1+x^2)=\kappa-1\qquad \text{so that } \eta^2=1.$$

Since the 'zero modes' are excluded from the propagator by the introduction of collective coordinates (§ 4), the propagator in coordinate space is given by

$$\tilde{V}^{-1}(\eta_1,\eta_2)=-\tilde{\sum_{L,\alpha}}\frac{Y_L^\alpha(\eta_1)Y_L^\alpha(\eta_2)^*}{(L+\frac{1}{2}d+2)(L+\frac{1}{2}d-3)}\tag{A.1}$$

where the tilde means $L=1$ is excluded and the eigenvalue in the denominator is read off from the L^2 eigenvalue in equation (10). From the Hamiltonian (22) the explicit expression for the graph is

$$6\epsilon \int \mathrm{d}\Omega_1 \, \kappa_1^{-1+(\epsilon/2)} \int \mathrm{d}\Omega_2 \, \kappa_2^{-\epsilon/2} \tilde{V}^{-1}(\eta_1, \eta_2) \tilde{V}^{-1}(\eta_2, \eta_2)$$

$$= 6\epsilon \int \mathrm{d}\Omega_1 \, \mathrm{d}\Omega_2 (1+\cos\theta_1)^{-1+(\epsilon/2)} (1+\cos\theta_2)^{-\epsilon/2}$$

$$\times \tilde{\sum_{L,L',\alpha,\alpha'}} \frac{Y_L^\alpha(\eta_1) Y_L^\alpha(\eta_2)^* Y_{L'}^{\alpha'}(\eta_2) Y_{L'}^{\alpha'}(\eta_2)^*}{(L+\frac{1}{2}d+2)(L+\frac{1}{2}d-3)(L'+\frac{1}{2}d+2)(L'+\frac{1}{2}d-3)} \tag{A.2}$$

(θ_1 and θ_2 are the two principal polar angles of η_1 and η_2). Since this expression has an ϵ factor from the $\hat{\Phi}$ vertex, we require only the $1/\epsilon$ pole from the ultraviolet divergence of the sum on L'. After some simplification using the result

$$\sum_{\alpha'} Y_{L'}^{\alpha'}(\eta_2) Y_{L'}^{\alpha'*}(\eta_2) = \frac{(2L'+d-1)\Gamma(\frac{1}{2}(d-1))\Gamma(d-1+L')}{4\pi^{(d+1)/2}\Gamma(L'+1)\Gamma(d-1)},$$

expression (A.2) reduces to

$$\tilde{\sum_{L,L',\alpha}} \frac{3\epsilon(2L'+d-1)\Gamma((\frac{1}{2}(d-1))\Gamma(d-1+L')}{2\pi(d+1)/2\Gamma(L'+1)\Gamma(d-1)}$$

$$\times \frac{\int \mathrm{d}\Omega_1 (1+\cos\theta_1)^{-1} Y_L^\alpha(\eta_1) \int \mathrm{d}\Omega_2 \, Y_L^\alpha(\eta_2)}{(L+\frac{1}{2}d+2)(L+\frac{1}{2}d-3)(L'+\frac{1}{2}d+2)(L'+\frac{1}{2}d-3)}$$

$$= -\frac{3\epsilon}{8} \tilde{\sum_{L'}} \frac{(2L'+d-1)\Gamma(d-1+L')}{\Gamma(L'+1)(L'+\frac{1}{2}d+2)(L'+\frac{1}{2}d-3)} (1+\mathrm{O}(\epsilon)).$$

(Note that only the bound state $L=0$ contributes.) Approximating the sum for large L', following the method of § 3, we obtain

$$\tilde{\sum_{L'}} \frac{(2L'+d-1)\Gamma(d-1+L')}{\Gamma(L'+1)(L'+\frac{1}{2}d+2)(L'+\frac{1}{2}d-3)} = \frac{12}{\epsilon} + \mathrm{O}(1).$$

Hence the graph has value $-\frac{9}{2}$ as stated in equation (15).

In § 4 on the introduction of collective coordinates we discussed how the combination of Jacobian and determinant factors is independent of the method of performing the calculation, i.e. independent of whether we work in flat space or on the sphere. Therefore we expect that the same value for this graph should be obtainable directly from the Hamiltonian (12) in flat space rather than (22) on the sphere. The expression corresponding to (A.2) is (to simplify we translate by x_0 and rescale by λ):

$$48\epsilon \int \mathrm{d}^d x \, \mathrm{d}^d y (1+x^2)^{-2} \tilde{M}^{-1}(x, y)(1+y^2)^{-1} \tilde{M}^{-1}(y, y). \tag{A.3}$$

We assume again we require only the $1/\epsilon$ pole from the divergent loop integral. It is useful in this context to view M_{yy}^{-1} as just the sum of free propagators with zero, one, two ... insertions of $-3g\phi_c^2(x) = 24(1+x^2)^{-2}$. The only *ultraviolet* contribution in dimensional regularisation is from the single insertion. Explicit evaluation of the singular term yields for expression (A.3)

$$\frac{144}{\pi^2} \int \frac{\mathrm{d}^d x \, \mathrm{d}^d y}{(1+x^2)^2} \tilde{M}^{-1}(x, y) \frac{1}{(1+y^2)^3} + \mathrm{O}(\epsilon).$$

This integral can be evaluated using the identity

$$M_{yz}\frac{1}{(1+z^2)^2}=\frac{4(d-6)}{(1+y^2)^3}$$

and the explicit form (16) for the dilatation and translation eigenfunctions. The result of these calculations is

$$\frac{144}{\pi^2}\frac{1}{4(d-6)}\int \mathrm{d}^d x\frac{1}{(1+x^2)^4}(1+\mathrm{O}(\epsilon))=-3+\mathrm{O}(\epsilon).$$

We have not been able to elucidate the discrepancy between these two calculations. The former result $-\frac{9}{2}$ is to be believed since the calculation is completely controlled; presumably some assumption in the second calculation is false, e.g. the zero modes may not have been correctly excluded.

Appendix 2

Here we summarise the generalisation of the calculations in §§ 2 to 6 to the case of $(\phi^2)^2$ interactions with $\mathrm{O}(n)$ internal symmetry.

The instanton solution of four dimensions is now

$$\phi_i=u_i\phi_c \qquad i=1,2,\ldots,n \tag{A.4}$$

where u_i is a unit vector and ϕ_c is as in equation (6). Since the Hamiltonian is $\mathrm{O}(n)$ invariant, the expression (13) for $H(\phi_c)$ is unchanged. The differential operator M_{ij} has the form

$$M_{ij}=\left(-\nabla^2-\frac{8\lambda^2}{[1+\lambda^2(x-x_0)^2]^2}\right)\delta_{ij}-\frac{16u_iu_j\lambda^2}{[1+\lambda^2(x-x_0)^2]^2}. \tag{A.5}$$

This operator can be decomposed into longitudinal and transverse components

$$M_{ij}=M_\mathrm{L}u_iu_j+M_\mathrm{T}(\delta_{ij}+u_iu_j). \tag{A.6}$$

Translating by x_0 and scaling by λ we have

$$M_\mathrm{L}=-\nabla^2-\frac{24}{(1+x^2)^2} \tag{A.7}$$

$$M_\mathrm{T}=-\nabla^2-\frac{8}{(1+x^2)^2}. \tag{A.8}$$

The determinant of M_L is as before (cf equation (23)). The determinant of the $(n-1)$-fold degenerate modes in M_T is obtained by the same procedure as in § 3. The equation analogous to (23) is

$$\left(\frac{\det M_\mathrm{T}}{\det M_{0\mathrm{T}}}\right)^{-(n-1)/2}=\prod_{L=0}^{\infty}\left(\frac{(L+\frac{1}{2}d-2)(L+\frac{1}{2}d+1)}{(L+\frac{1}{2}d-1)(L+\frac{1}{2}d)}\right)^{-\frac{1}{2}(n-1)\nu_\mathrm{L}(d+1)}. \tag{A.9}$$

In the numerator on the right-hand side we see the $L=0$ zero mode with eigenvalue

$$E=-\tfrac{3}{2}\epsilon(1+\mathrm{O}(\epsilon)) \tag{A.10}$$

corresponding to the spontaneous breaking of the $\mathrm{O}(n)$ symmetry. Extracting this factor from the product (A.9), and following the same procedure as in equations (25) to

(27), yields an additional factor

$$c_2^{(n)}=2^{(n-1)/2}\prod_{L=1}^{\infty}\left(\frac{(L+\frac{1}{2}d-2)(L+\frac{1}{2}d+1)}{(L+\frac{1}{2}d-1)(L+\frac{1}{2}d)}\right)^{-\frac{1}{2}(n-1)\nu_L(d+1)}$$

$$=(2^{-1/6}\pi^{-1/6}3^{1/2})^{n-1}\exp\left[(n-1)\left(\frac{1}{3\epsilon}+\frac{1}{4}-\frac{1}{2}\gamma+\frac{\zeta'(2)}{\pi^2}\right)(1+\mathrm{O}(\epsilon))\right]. \quad (\mathrm{A}.11)$$

The $(n-1)$-fold zero modes (A.10) are replaced by collective coordinate integrations over the unit vectors $\{u_i\}$. Following the arguments in § 4, the appropriate Jacobian factor is

$$J^{V_{\mathrm{T}}}=\left(\int \mathrm{d}^d x\, \phi_c(x)\frac{4\lambda^2}{[1+\lambda^2(x-x_0)^2]^2}\phi_c(x)\right)^{(n-1)/2}=\left(-\frac{16\pi^2\lambda^\epsilon}{3g}\right)^{(n-1)/2}. \quad (\mathrm{A}.12)$$

The final correction factor for the bare theory is the evaluation of figure 1. Following appendix 1, one obtains the value $-(n+8)/2$. Thus for $n\neq 1$ this graph contributes a factor

$$c_1^{(n)}=\exp[-(n-1)/2] \quad (\mathrm{A}.13)$$

in addition to the factor $c_1=\exp(-\frac{9}{2})$ in equation (15).

Combining expressions (A.11) to (A.13) and remembering the factors $(2\pi)^{-(n-1)/2}$ for the Gaussians replaced by collective coordinates, one obtains the *additional* factor

$$\left(-\frac{8\pi^2\lambda^\epsilon}{3g}\right)^{(n-1)/2}(3^{1/2}\pi^{-2/3}2^{-1/6})^{(n-1)}\exp(n-1)\left(\frac{1}{3\epsilon}+\frac{1}{\pi^2}\zeta'(2)-\frac{1}{4}-\frac{1}{2}\gamma\right)$$
$$\times\frac{1}{2}\int \mathrm{d}\boldsymbol{u}\, u_{i_1}u_{i_2}\ldots u_{i_{2M}}\Big] \quad (\mathrm{A}.14)$$

for $n\neq 1$, for the imaginary part of the $2M$-point vertex function. Note that we have put in a factor of $\frac{1}{2}$ since, according to the discussion of the integral (17), there is half a Gaussian integral for the bound state at each saddle point. Expression (A.14) can be simplified using the result

$$\frac{1}{2}\int \mathrm{d}\boldsymbol{u}\, u_{i_1}u_{i_2}\ldots u_{i_{2M}}=\frac{\pi^{n/2}}{\Gamma(n/2)}\frac{(\delta_{i_1i_2}\delta_{i_3i_4}\ldots\delta_{i_{2M-1}i_{2M}}+\text{perms})}{n(n+2)\ldots(n+2M-2)}. \quad (\mathrm{A}.15)$$

It is seen explicitly that (A.14) is 1 for $n=1$.

Expression (A.14) contains a divergent term which is again removed by coupling constant renormalisation; for $n\neq 1$, equation (41) becomes

$$\frac{\lambda^\epsilon}{g}=\frac{1}{g_{\mathrm{R}}(\lambda)}-\frac{n+8}{8\pi^2\epsilon}. \quad (\mathrm{A}.16)$$

Making the substitution for g yields

$$\operatorname*{Im}_{(\arg g=\pi)}\Gamma_{\mathrm{R}}^{(2M)}(q_i)$$
$$=-C_{\mathrm{R}}\int_0^\infty\frac{\mathrm{d}\lambda}{\lambda}\lambda^{d-M(d-2)}\left(-\frac{8\pi^2}{3g_{\mathrm{R}}(\lambda)}\right)^{(d+n+2M)/2}\prod_{i=1}^{2M}\frac{q_i^2}{\lambda^2}\tilde{\phi}\left(\frac{q_i}{\lambda}\right)$$
$$\times\exp\left[\frac{8\pi^2}{3g_{\mathrm{R}}(\lambda)}\left(1-\frac{\epsilon}{2}(\ln\pi+\gamma+2)\right)\right]\frac{(\delta_{i_1i_2}\delta_{i_3i_4}\ldots\delta_{i_{2M-1}i_{2M}}+\text{perms})}{1.3.\ldots.(2M-1)}$$
$$\times(1+\mathrm{O}(\epsilon,g)) \quad (\mathrm{A}.17)$$

where

$$C_R = 2^{-1/2}\pi^{-3/2} \exp\left(\frac{3}{\pi^2}\zeta'(2) - 2\gamma - \frac{3}{4}\right)(3^{1/2}2^{-1/6})^{(n-1)}\frac{\Gamma(M+\frac{1}{2})}{\Gamma(M+\frac{1}{2}n)}$$
$$\times \exp[(n-1)(\pi^{-2}\zeta'(2) - \tfrac{1}{3}\gamma + \tfrac{1}{12})]. \tag{A.18}$$

This generalises the results (42) and (43) for $n \neq 1$.

The calculations to obtain late terms in the β function follow § 6. The expression generalising (48) is

$$\tilde{\beta}_K = \frac{C_R}{2\pi}\left(-\frac{3}{8\pi^2}\right)^K \Gamma(K+4+\tfrac{1}{2}n)\left(\frac{\mu}{\mu_0}\right)^{-(n+8)/3}\frac{n+8}{9}$$
$$\times \int_0^\infty dx\, x^{(n+8)/3+7}(\tilde{\phi}(x))^4(1+O(K^{-1})) \tag{A.19}$$

where C_R is given in (A.18). The multiplicity of the one-loop graph again modifies the coefficient α in expression (52):

$$\alpha = \frac{n+8}{8\pi^2}(\tfrac{1}{2}\ln\pi + \tfrac{1}{2}\ln 3 - \tfrac{1}{2}\gamma + 1 - \ln(\mu/\mu_0)]. \tag{A.20}$$

The final result for β_K becomes

$$\beta_K = 3^{1/2}2^{13/2}\exp\left(\frac{3}{\pi^2}\zeta'(2) - \frac{15}{4} - \frac{1}{2}\gamma\right)\frac{n+8}{9}\frac{\Gamma(\frac{5}{2})}{\Gamma(2+\frac{1}{2}n)}\left(\frac{9}{2\pi}\right)^{(n-1)/6}$$
$$\times \exp(n-1)(\pi^{-2}\zeta'(2) - \tfrac{1}{4} - \tfrac{1}{6}\gamma)\int_0^\infty dx\, x^{(m+8)/3+3}(K_1(x))^4$$
$$\times\left(-\frac{3}{8\pi^2}\right)^K \Gamma(K+4+\tfrac{1}{2}n)(1+O(K^{-1}). \tag{A.21}$$

Using the expression (cf § 6)

$$\sum_{l=2}^{\infty}(l+1)(l+2)(2l+3)\left[\ln\left(1-\frac{2}{(l+1)(l+2)}\right)+\frac{2}{(l+1)(l+2)}+\frac{2}{(l+1)^2(l+2)^2}\right]$$
$$= -12\pi^{-2}\zeta'(2) + 6\gamma - 24\ln 2 + 24\ln 3 + 2\ln 2\pi - \tfrac{55}{3}$$

one may verify that the expression (A.21) agrees with the result of Brézin *et al* (1977). We find however that there is an error in the numerical values they quote, as has also been noted by Dittes *et al* (1977) who compare explicit four-loop calculations with the estimate (A.21).

Appendix 3

Here we make some remarks on conformal invariant $g\phi^{2N}$ theories ($N > 2$). These interactions have a dimensionless coupling constant in $d_c = 2N/(N-1)$ dimensions, and are conformal invariant in that dimension if there is no mass term or other interaction. In this case there exist analytic instanton solutions which again enable high-order estimates to be made (Lipatov 1977a, b, Brézin *et al* 1977).

These calculations are particularly simple in dimensional regularisation because for $N > 2$ no renormalisation is required at the one-loop level; to be precise, graphs of the form figure 4 vanish in the dimensional regularisation scheme when the propagator is

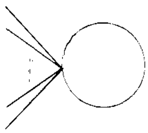

Figure 4. Graph contributing to $\Gamma^{(2N-2)}$ which vanishes in the massless theory in dimensional regularisation.

massless. Thus the imaginary part calculated as in §§ 2 to 4 should also be finite for these theories. This is obvious for the case $N > 3$; because the conformal dimension d_c is not an integer, there are no divergences in the limit $d \to d_c$. The asymptotic behaviour ($L \to \infty$) of the degeneracy (11)

$$\nu_L(d+1) = \frac{(2L+d-1)\Gamma(L+d-1)}{\Gamma(d)\Gamma(L+1)} \sim \frac{2}{\Gamma(d)} L^{d-1}(1+O(L^{-1}))$$

ensures that for non-integer d one never meets the pole of the Riemann ζ function. For the case $N = 3$, one may verify explicitly that in the sum

$$\sum_L \nu_L(d+1) \ln(E^V(L)/E^{V_0}(L))$$

(where $E^V(L)$ ($E^{V_0}(L)$) are the eigenvalues in the presence (absence) of an instanton), there is no term of the form $\Sigma_L 1/L^{1+\epsilon}$ for L large; its coefficient vanishes.

We leave the reader to construct the result from the expressions in Lipatov (1977a, b) and Brézin *et al* (1977).

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